PERGAMON

International Journal of Solids and Structures 36 (1999) 3391–3402

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

Vibrations of multilayer plates under the effect of impulse loads. Three-dimensional theory

A. N. Shupikov*, S. V. Ugrimov

Institute for Problems in Machinery of the Ukrainian National Academy of Sciences, 2/10 Pozharsky St, Kharkov 310046, Ukraine

Received 30 January 1998; in revised form 8 April 1998

Abstract

A new analytical–numerical approach to investigation of the response of multilayer plates to impulse loading is described in this paper. The plate's behaviour is described by the equations of the three-dimensional elasticity theory. According to the approach being proposed, the sought for functions included in the system of equations and the boundary and initial conditions are presented as Fourier series expansions in the tangential directions. The derivatives of these functions in the transverse direction are replaced by their finite-difference presentations. As a result of such transforms, the problem of vibration of a multilayer plate is reduced to integration of a system of ordinary differential equations with constant coefficients. Integration is performed by expansion into the Taylor's series. The possibilities of the approach proposed and the validity of results obtained is illustrated by several examples of calculating vibration processes and the processes of propagation of elastic waves. A comparison of the results obtained on the basis of other approaches has been performed.

1. Introduction

As a rule, two-dimensional theories (see, for example, Grigoliuk and Kogan, 1972; Grigoliuk and Kulikov, 1988; Reddy, 1989, 1993; Smetankina et al., 1995) are used for investigating the response of multilayer structures. In this case, it is relatively easy to obtain analytical solutions and numerical results. Within the framework of two-dimensional theories, the stressed–strained state is described approximately. In some cases the use of these theories yields invalid results (e.g. big relative thickness, necessity of investigating wave processes, etc.).

The behaviour of multilayer structures is described most accurately within the framework of the three-dimensional elasticity theory. In so doing, obtaining analytical solutions and numerical results are connected with overcoming significant difficulties.

* Corresponding author. Fax: 00 380 0572 944 635. E-mail: kantor@ipmach.kharkov.ua

0020–7683/99/$ - see front matter
PII: S0020–7683(98)00156–5

In this statement the many problems have been solved for the case of static loads (Pagano, 1969; Little, 1973).

In this work, an analytical–numerical method of solving the problem of transient vibrations of a multilayer plate under impulse loading has been suggested within the framework of the three-dimensional elasticity theory. The behaviour of the plate is described by Lamé's equations.

The essence of the method consists in that the sought for displacement functions are presented in the form of expansions into double Fourier series in the direction of plane coordinates x_1, x_2 up to a complete orthogonal system of trigonometric functions satisfying the boundary conditions over the plate contour. The partial derivatives of these functions in the direction of the normal coordinate x_3 are replaced by their finite-difference presentations. As a result of these transforms, the problem of transient vibration of a multilayer plate under the action of an impulse load is reduced to integration of a system of ordinary differential equations with constant coefficients. The system is integrated by expanding the solution into the Taylor's series (Bakhvalov, 1975).

The method described in this paper allows one to obtain the solution of the dynamic problem and numerical results relatively simply.

The feasibility of the method and the validity of results are demonstrated by several examples. In the case of a static load, the results obtained by the method described are compared with the exact solution (Little, 1973) as well as with the results obtained on the basis of the first-order (Reissner, 1944) and high-order (Lo et al., 1977) two-dimensional theories. The first-order theory is based on the following kinematic hypotheses

$$\begin{aligned} u &= u_0 + u_1 \cdot x_3 \\ v &= v_0 + v_1 \cdot x_3 \\ w &= w_0, \end{aligned} \tag{1}$$

the high-order theory is based on the following relationships

$$\begin{aligned} u &= u_0 + u_1 \cdot x_3 + u_2 \cdot x_3^2 + u_3 \cdot x_3^3 \\ v &= v_0 + v_1 \cdot x_3 + v_2 \cdot x_3^2 + v_3 \cdot x_3^3 \\ w &= w_0 + w_1 \cdot x_3 + w_2 \cdot x_3^2, \end{aligned} \tag{2}$$

where $u = u(x_1, x_2, x_3)$, $v = v(x_1, x_2, x_3)$—displacements of the plate point in the direction of the plane coordinates x_1 and x_2, correspondingly; $w = w(x_1, x_2, x_3)$—displacement of the plate point in the direction of the normal coordinate x_3.

When studying transient vibrations of a three-layer plate under the action of impulse loading, the results are compared with similar data obtained with the help of a rectified high-order theory (Shupikov and Ugrimov, 1997).

The phenomena of propagation and reflection of elastic waves are investigated by example of a one- and two-layer plate.

2. Problem statement

A multilayer rectangular simple supported plate consists of I homogeneous isotropic layers of constant thickness. The geometric parameters of the pack are as follows: A, B are the plate plan

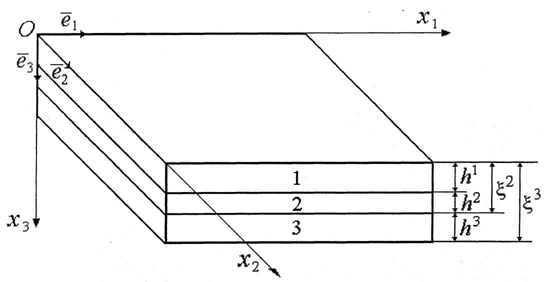

Fig. 1. Multilayer plate.

dimensions; h^i is the thickness of the i-th layer. It is assumed that contact between layers excludes their delamination and mutual slipping. The transverse load $q = q(x_1, x_2, t)$ is applied to the external surface of the first layer.

The plate is referred to the Cartesian system of coordinates $Ox_1x_2x_3$ and the coordinate plane Ox_1x_2 is linked with the external surface of the first layer (Fig. 1). The vectors $\bar{e}_1$, $\bar{e}_2$, $\bar{e}_3$ are unit vectors of axes Ox_1, Ox_2, Ox_3 correspondingly.

The behaviour of each layer is described by Lamé's equations (see Novatsky, 1975):

$$\mu^i(\bar{\nabla}\cdot\bar{\nabla})\bar{u}^i+(\lambda^i+\mu^i)\bar{\nabla}(\bar{\nabla}\cdot\bar{u}^i) = \rho^i\frac{\partial^2\bar{u}^i}{\partial t^2}, \tag{3}$$

where $\bar{\nabla}$ is gradient, i.e.

$$\bar{\nabla} = \bar{e}_1\frac{\partial}{\partial x_1}+\bar{e}_2\frac{\partial}{\partial x_2}+\bar{e}_3\frac{\partial}{\partial x_3}.$$

The system of eqns (3) is solved in combination with the conditions on the external surfaces of the 1-st and I-th layers

$$p^1_{31} = p^1_{32} = 0, \quad p^1_{33} = -q \quad \text{at } x_3 = 0;$$

$$p^I_{3k} = 0 \quad \text{at } x_3 = \xi^I, \quad k = 1,2,3,$$

$$\xi^i = \sum_{j=1}^{i} h^j, \tag{4}$$

boundary conditions over the plate contour

$$p^i_{11} = u^i_2 = u^i_3 = 0, \quad \text{at } x_1 = 0, A;$$

$$p^i_{22} = u^i_1 = u^i_3 = 0, \quad \text{at } x_2 = 0, B, \quad i = \overline{1,I}; \tag{5}$$

contact conditions of adjacent layers

$$u^i_k = u^{i+1}_k, \quad p^i_{3k} = p^{i+1}_{3k}, \quad \text{at } x_3 = \xi^i, \quad k = 1,2,3, \quad i = \overline{1,I-1}; \tag{6}$$

and initial conditions

$$\bar{u}^i(x_1, x_2, x_3, 0) = \frac{\partial \bar{u}^i(x_1, x_2, x_3, 0)}{\partial t} = 0, \quad i = \overline{1, I}. \tag{7}$$

Here i is the layer number; λ^i, μ^i are Lamé's coefficients; ρ^i is the specific density; $\bar{u}^i$ is the displacement vector of the i-th layer point; $u_k^i = u_k^i(x_1, x_2, x_3, t)$ is the projection of the displacement vector on the coordinate axis Ox_k $(k = 1, 2, 3)$; p_{jk}^i are the components of the stress tensor.

The stress tensor components are calculated by the formulae

$$p_{jk}^i = \lambda^i \delta_{jk} u_{ll}^i + 2\mu^i u_{jk}^i,$$

$$\delta_{jk} = \begin{Bmatrix} 0, & j \neq k \\ 1, & j = k \end{Bmatrix}, \quad u_{ll}^i = u_{11}^i + u_{22}^i + u_{33}^i, \quad i = \overline{1, I}, \quad j, k = 1, 2, 3, \tag{8}$$

where u_{jk}^i is the strain determined by Cauchy's relationships

$$u_{jk}^i = \frac{1}{2}\left(\frac{\partial u_j^i}{\partial x_k} + \frac{\partial u_k^i}{\partial x_j}\right). \tag{9}$$

Lamé's coefficients are linked with Young's modulus (E^i) and Poisson's coefficient (v^i) by the relationship

$$\lambda^i = \frac{v^i E^i}{(1+v^i)(1-2v^i)}, \quad \mu^i = \frac{E^i}{2(1+v^i)}.$$

3. Solution method

Displacement and the external load are expanded into double Fourier series by a system of orthogonal functions satisfying their boundary conditions (5)

$$u_k^i = \sum_{m=1}^{M} \sum_{n=1}^{N} \Phi_{kmn}^i(x_3, t) \cdot B_{kmn}(x_1, x_2), \quad k = 1, 2, 3, \quad i = \overline{1, I};$$

$$q = \sum_{m=1}^{M} \sum_{n=1}^{N} q_{mn}(x_3, t) \cdot B_{3mn}(x_1, x_2),$$

$$B_{1mn} = \cos\frac{m\pi x_1}{A}\sin\frac{n\pi x_2}{B}, \quad B_{2mn} = \sin\frac{m\pi x_1}{A}\cos\frac{n\pi x_2}{B}, \quad B_{3mn} = \sin\frac{m\pi x_1}{A}\sin\frac{n\pi x_2}{B}. \tag{10}$$

The partial derivatives of the functions $\Phi_{kmn}^i(x_3, t)$ over coordinate x_3 are replaced by their finite-difference presentations. For this, in each layer a regular grid is built

$$x_3^{i(s)} = \xi^{i-1} + s\tau^i, \quad s = \overline{0, S^i}, \quad \tau^i = \frac{h^i}{S^i}, \quad i = \overline{1, I}.$$

Let us define

$$\Phi_{kmn}^{i(s)} = \Phi_{kmn}^i(x_3^{i(s)}, t).$$

For approximation of partial derivatives, a three-point template is used (Forsythe and Wasov, 1960)

$$\frac{\partial \Phi_{kmn}^{i(s)}}{\partial x_3} = \frac{\Phi_{kmn}^{i(s+1)} - \Phi_{kmn}^{i(s-1)}}{2\tau^i}, \quad \frac{\partial^2 \Phi_{kmn}^{i(s)}}{\partial x_3^2} = \frac{\Phi_{kmn}^{i(s+1)} - 2\Phi_{kmn}^{i(s)} + \Phi_{kmn}^{i(s-1)}}{(\tau^i)^2}. \tag{11}$$

As a result of these transforms, system (3) for each pair (m, n) takes the form

$$\begin{aligned}
&\frac{\mu^i}{(\tau^i)^2}\Phi_{1mn}^{i(s-1)} - \left[\frac{2\mu^i}{(\tau^i)^2} + \mu^i \frac{n^2\pi^2}{B^2} + (\lambda^i + 2\mu^i)\frac{m^2\pi^2}{A^2}\right]\Phi_{1mn}^{i(s)} + \frac{\mu^i}{(\tau^i)^2}\Phi_{1mn}^{i(s+1)} \\
&-(\lambda^i + \mu^i)\frac{mn\pi^2}{AB}\Phi_{2mn}^{i(s)} + \frac{\lambda^i + \mu^i}{2\tau^i}\frac{m\pi}{A}(\Phi_{3mn}^{i(s+1)} - \Phi_{3mn}^{i(s-1)}) = \rho^i \frac{\mathrm{d}^2\Phi_{1mn}^{i(s)}}{\mathrm{d}t^2}, \\
&-(\lambda^i + \mu^i)\frac{mn\pi^2}{AB}\Phi_{1mn}^{i(s)} + \frac{\mu^i}{(\tau^i)^2}\Phi_{2mn}^{i(s-1)} - \left[\frac{2\mu^i}{(\tau^i)^2} + \mu^i\frac{m^2\pi^2}{A^2} + (\lambda^i + 2\mu^i)\frac{n^2\pi^2}{B^2}\right]\Phi_{2mn}^{i(s)} \\
&+\frac{\mu^i}{(\tau^i)^2}\Phi_{2mn}^{i(s+1)} + \frac{\lambda^i + \mu^i}{2\tau^i}\frac{n\pi}{B}(\Phi_{3mn}^{i(s+1)} - \Phi_{3mn}^{i(s-1)}) = \rho^i\frac{\mathrm{d}^2\Phi_{2mn}^{i(s)}}{\mathrm{d}t^2}, \\
&-\frac{\lambda^i + \mu^i}{2\tau^i}\frac{m\pi}{A}(\Phi_{1mn}^{i(s+1)} - \Phi_{1mn}^{i(s-1)}) - \frac{\lambda^i + \mu^i}{2\tau^i}\frac{n\pi}{B}(\Phi_{2mn}^{i(s+1)} - \Phi_{2mn}^{i(s-1)}) + \frac{\lambda^i + 2\mu^i}{(\tau^i)^2}\Phi_{3mn}^{i(s-1)} \\
&-\left[\frac{2\lambda^i + 4\mu^i}{(\tau^i)^2} + \mu^i\left(\frac{m^2\pi^2}{A^2} + \frac{n^2\pi^2}{B^2}\right)\right]\Phi_{3mn}^{i(s)} + \frac{\lambda^i + 2\mu^i}{(\tau^i)^2}\Phi_{3mn}^{i(s+1)} = \rho^i\frac{\mathrm{d}^2\Phi_{3mn}^{i(s)}}{\mathrm{d}t^2},
\end{aligned} \tag{12}$$

$s = \overline{0, S^i}$, $i = \overline{1, I}$.

Conditions (4), (6) and (7) are defined by the expressions

$$\begin{aligned}
&\frac{\Phi_{1mn}^{1(1)} - \Phi_{1mn}^{1(-1)}}{2\tau^1} + \frac{m\pi}{A}\Phi_{3mn}^{1(0)} = 0, \quad \frac{\Phi_{2mn}^{1(1)} - \Phi_{2mn}^{1(-1)}}{2\tau^1} + \frac{n\pi}{B}\Phi_{3mn}^{1(0)} = 0, \\
&-\lambda^1\left(\frac{m\pi}{A}\Phi_{1mn}^{1(0)} + \frac{n\pi}{B}\Phi_{2mn}^{1(0)}\right) + (\lambda^1 + 2\mu^1)\frac{\Phi_{3mn}^{1(1)} - \Phi_{3mn}^{1(-1)}}{2\tau^1} = -q_{mn}, \\
&\frac{\Phi_{1mn}^{I(S^I+1)} - \Phi_{1mn}^{I(S^I-1)}}{2\tau^I} + \frac{m\pi}{A}\Phi_{3mn}^{I(S^I)} = 0, \quad \frac{\Phi_{2mn}^{I(S^I+1)} - \Phi_{2mn}^{I(S^I-1)}}{2\tau^I} + \frac{n\pi}{B}\Phi_{3mn}^{I(S^I)} = 0, \\
&-\lambda^I\left(\frac{m\pi}{A}\Phi_{1mn}^{I(S^I)} + \frac{n\pi}{B}\Phi_{2mn}^{I(S^I)}\right) + (\lambda^I + 2\mu^I)\frac{\Phi_{3mn}^{I(S^I+1)} - \Phi_{3mn}^{I(S^I-1)}}{2\tau^I} = 0;
\end{aligned} \tag{13}$$

$$\Phi_{kmn}^{i(S^i)} = \Phi_{kmn}^{i+1(0)}, \quad k = 1, 2, 3,$$

$$\mu^i\left(\frac{\Phi_{1mn}^{i(S^i+1)}-\Phi_{1mn}^{i(S^i-1)}}{2\tau^i}+\frac{m\pi}{A}\Phi_{3mn}^{i(S^i)}\right)=\mu^{i+1}\left(\frac{\Phi_{1mn}^{i+1(1)}-\Phi_{1mn}^{i+1(-1)}}{2\tau^{i+1}}+\frac{m\pi}{A}\Phi_{3mn}^{i+1(0)}\right),$$

$$\mu^i\left(\frac{\Phi_{2mn}^{i(S^i+1)}-\Phi_{2mn}^{i(S^i-1)}}{2\tau^i}+\frac{n\pi}{B}\Phi_{3mn}^{i(S^i)}\right)=\mu^{i+1}\left(\frac{\Phi_{2mn}^{i+1(1)}-\Phi_{2mn}^{i+1(-1)}}{2\tau^{i+1}}+\frac{n\pi}{B}\Phi_{3mn}^{i+1(0)}\right),$$

$$-\lambda^i\left(\frac{m\pi}{A}\Phi_{1mn}^{i(S^i)}+\frac{n\pi}{B}\Phi_{2mn}^{i(S^i)}\right)+(\lambda^i+2\mu^i)\frac{\Phi_{3mn}^{i(S^i+1)}-\Phi_{3mn}^{i(S^i-1)}}{2\tau^i}$$

$$=-\lambda^{i+1}\left(\frac{m\pi}{A}\Phi_{1mn}^{i+1(0)}+\frac{n\pi}{B}\Phi_{2mn}^{i+1(0)}\right)+(\lambda^{i+1}+2\mu^{i+1})\frac{\Phi_{3mn}^{i+1(1)}-\Phi_{3mn}^{i+1(-1)}}{2\tau^{i+1}},\quad i=\overline{1,I-1};\qquad(14)$$

$$\Phi_{kmn}^{i(s)}(0)=\frac{\mathrm{d}\Phi_{kmn}^{i(s)}(0)}{\mathrm{d}t}=0,\quad i=\overline{1,I}.\qquad(15)$$

Conditions (5) are met exactly by a corresponding selection of the coordinate functions B_{kmn}.

Conditions (13) and (14) allow one to exclude the values $\Phi_{kmn}^{i(-1)}$, $\Phi_{kmn}^{i(S^i+1)}$ ($i=\overline{1,I}; k=1,2,3; m=\overline{1,M}; n=\overline{1,N}$) of the sought for functions in the "extracontour" points from system (12).

Hence, the solution of problems (3)–(7) on the dynamic response of a multilayer plate to action of an impulse load is reduced for each pair (m,n) to integration of a system of ordinary differential equations with constant coefficients. In this paper, the system obtained is integrated by using the method of expanding the solution into a Taylor series.

4. Numerical results

The feasibility of the method and the validity of results may be illustrated by several examples.

In case of static loading of an infinite homogeneous strip ($h/A=1.5, v=0.25$), the results of calculations according to the method described are compared with exact solutions given in the paper by Little (1973), as well as with data obtained from the two-dimensional theories (Lo et al., 1977). The strip is effected by a load

$$q(x_1)=q_0\sin\frac{\pi x_1}{A}.$$

Figure 2 shows the stress distribution over the plate thickness in its middle section.

The possibility of investigating wave and vibration processes is demonstrated for impulse loading. One-, two- and three-layer plates are considered here. The plate parameters are given in Table 1.

One- and two-layer plates are effected by the load

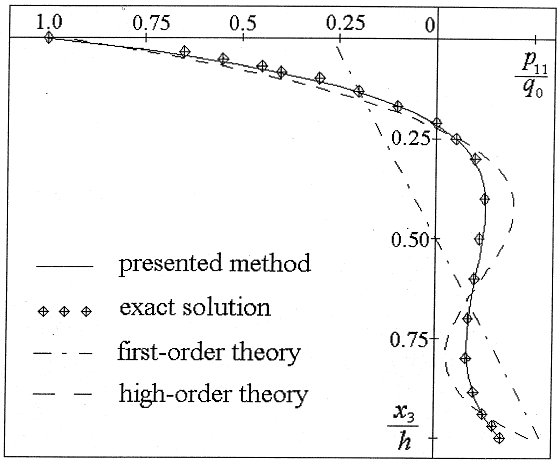

Fig. 2. Distribution of stresses over the thickness in a strip middle section.

Table 1
Parameters of multilayer plates

Plate	A (m)	B (m)	i	h^i (m)	E^i (MPa)	ν^i	ρ^i (kg/m^3)
One-layer	0.3	0.3	1	0.1	6.67×10^4	0.22	2.5×10^3
Two-layer	0.3	0.3	1	0.05	5.59×10^3	0.38	1.2×10^3
			2	0.05	6.67×10^4	0.22	2.5×10^3
Three-layer	0.42	0.47	1	0.012	6.67×10^4	0.22	2.5×10^3
			2	0.002	2.74×10^2	0.38	1.2×10^3
			3	0.012	6.67×10^4	0.22	2.5×10^3

$$q(x_1,x_2,t) = q_0 \exp\left(-\frac{t}{T}\right)\sin\frac{\pi x_1}{A}\sin\frac{\pi x_2}{B},$$

where $q_0 = 100$ kPa and $T = 28$ μs.

Figure 3 shows the dynamic response in the middle of a homogeneous plate. Data were obtained for $S^1 = 100$. The interval t_1 corresponds to the time of travel of a dilatational wave over the plate thickness and it corresponds to the similar value obtained from the precise formula

$$t_1 = \frac{h^1}{V} \approx 18.1\ \mu\text{s},$$

where V is the dilatational wave velocity (Novatsky, 1975)

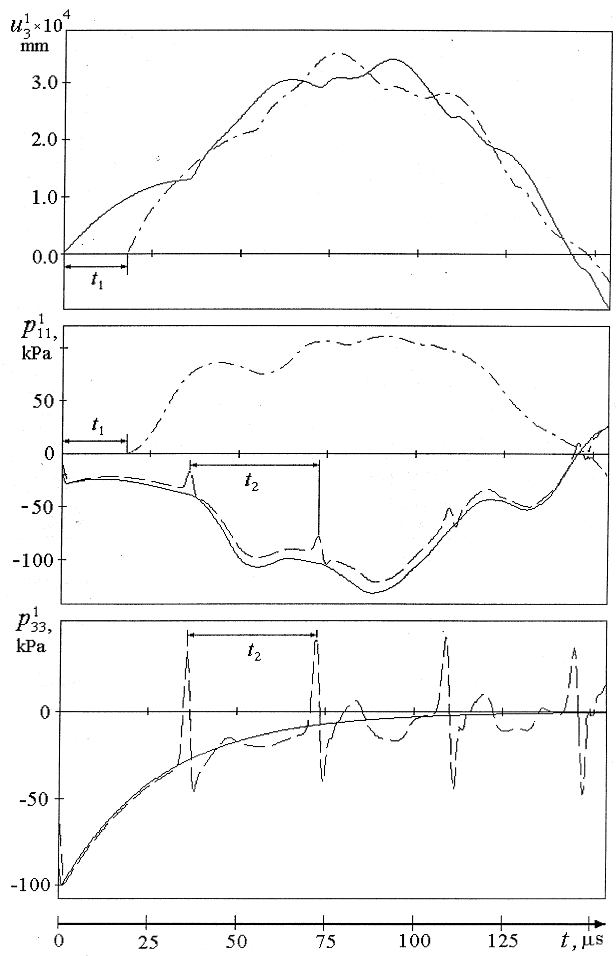

Fig. 3. One-layer plate response: solid line, $x_3 = 0$; dashed line, $x_3 = 5$ mm; sloping dash–dot line, $x_3 = 100$ mm.

$$V = \sqrt{\frac{\lambda^1 + 2\mu^1}{\rho^1}} \approx 5.52 \cdot 10^3 \text{ m/s}.$$

Periodic "surges" seen on the graphs for stresses p_{11}^1 and p_{33}^1 (dashed line) appear in those moments of time when the wave reflected from the external surface ($x_3 = h^1$) arrives at the point being considered ($x_3 = 5$ mm). Period t_2 corresponds to the time the wave travels the distance $l = 2(h^1 - 0.005)$ m and coincides with the same value obtained from the precise formula

$$t_2 = \frac{l}{V} \approx 34.4\ \mu\text{s}.$$

Figures 4a and b show the dynamic response in the middle of a two-layer plate. Data were obtained for $S^1 = S^2 = 50$. The pattern of the relationships shown in Figs 4 is the same as in Fig. 3 for a homogeneous plate. In the case of the "surges" of stresses p_{11}^i, p_{33}^i $(i = 1, 2)$ appear at the

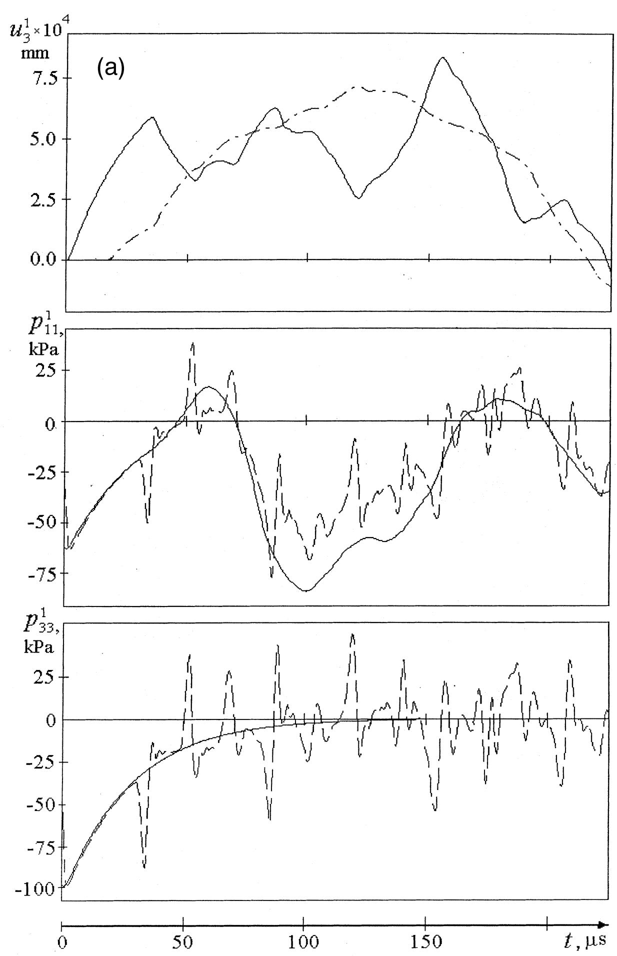

Fig. 4a. Two-layer plate response. First layer: solid line, $x_3 = 0$; dashed line, $x_3 = 5$ mm; sloping dash–dot line, $x_3 =$ 50 mm.

moment of time when the waves reflected from the layers' interfaces or from the external surfaces of the plate arrive at the point being considered ($x_3 = 5$ mm).

The vibrations of a three-layer rectangular plate have been investigated for the action of an evenly distributed impulse load

$$q(x_1, x_2, t) = q_0 H(t), \quad 0 \leqslant x_1 \leqslant A, \quad 0 \leqslant x_2 \leqslant B,$$

where $H(t)$ is Heaviside's function, $q_0 = 10$ kPa.

Figures 5 and 6 show the time dependence of deflections and stresses in the plate middle section on the external surface of the third layer. Data were obtained for $S^1 = S^3 = 10$, $S^2 = 8$, $M = N = 25$. The results of calculations by the method proposed are compared with similar data obtained on the basis of a high-order two-dimensional theory (Shupikov and Ugrimov, 1997), which is founded on hypothesis (2) for each layer.

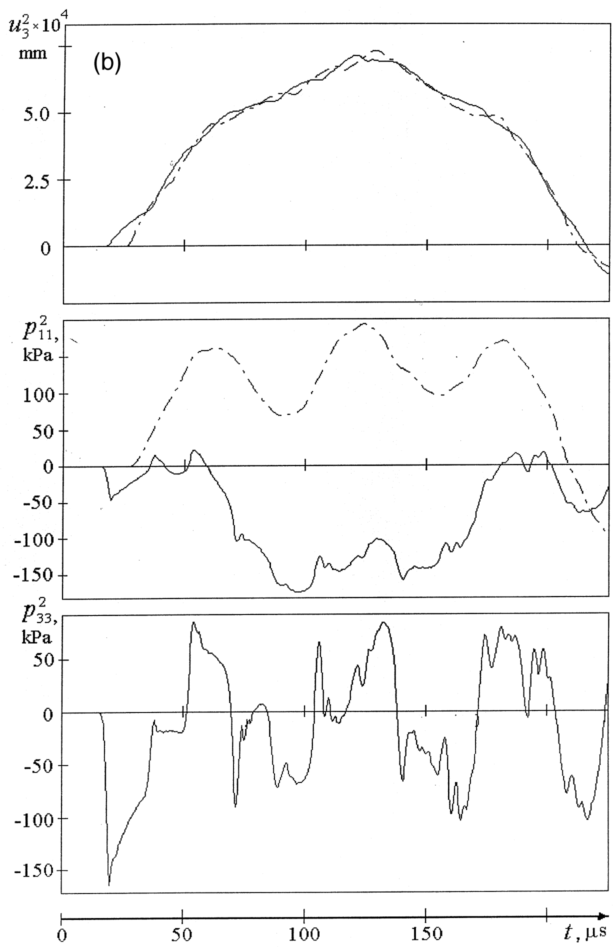

Fig. 4b. Two-layer plate response. Second layer: solid line, $x_3 = 50$ mm; sloping dash–dot line, $x_3 = 100$ mm.

5. Conclusions

A simple to implement analytical–numerical method of solving the problem of transient vibrations of multilayer plates within the framework of the three-dimensional elasticity theory has been proposed.

Calculation results (Figs 2, 5 and 6) demonstrate a fine matching of results obtained on the basis of the method proposed with the exact solution and calculation results based on the high-order two-dimensional theory.

The possibility of investigating wave processes in thick homogeneous and multilayer plates has been demonstrated. The plotted relationships (Figs 3 and 4) demonstrate an express wave character.

The method proposed may be useful for assessing the field of application of two-dimensional theories when it is necessary to investigate the process of elastic wave propagation, as well as when

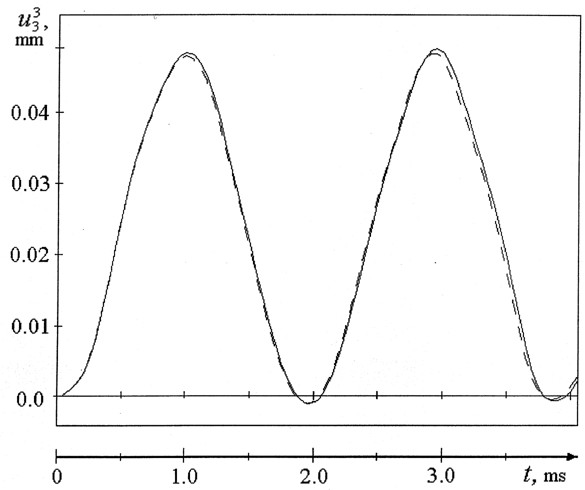

Fig. 5. Deflections of external surfaces of three-layer plate under impulse loading: solid line, presented method; dashed line, high-order two-dimensional theory.

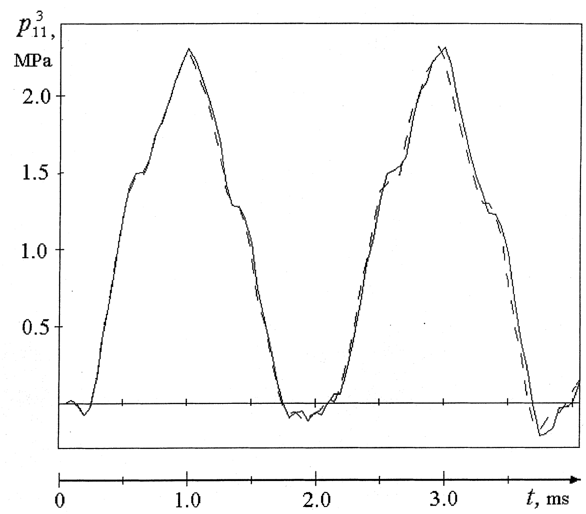

Fig. 6. Stresses of external surfaces of three-layer plate under impulse loading: solid line, presented method; dashed line, high-order two-dimensional theory.

the stressed–strained state of a structure being investigated has an essentially three-dimensional character.

The application of the method being considered is not limited by the case of isotropic plates. It may well be extended to the case of anisotropic plates.

References

Bakhvalov, N.S., 1975. Numerical Methods. Nauka, Moscow.

Forsythe, G.E., Wasov, W.R., 1960. Finite-Difference Methods for Partial Differential Equations. John Wiley and Sons, New York, London.

Grigoliuk, E.I., Kogan, F.A., 1972. Modern state of the theory of multilayer shells. Prikladnaya Mekhanika 8(6), 3–17.

Grigoliuk, E.I., Kulikov, G.M., 1988. General direction of development of the theory of multilayer shells. Mekhanika Kompozitnykh Materialov 24(2), 287–298; English translation in Mechanics of Composite Materials 24(2), 231–241.

Little, R.W., 1973. Elasticity. Prentice-Hall, Englewood Cliffs, N.J.

Lo, K.H., Christensen, R.M., Wu, E.M., (1977). A high-order theory of plate deformation. Journal of Applied Mechanics 44, 663–676.

Novatsky, V., 1975. Theory of Elasticity. Mir, Moscow.

Pagano, N.J., 1969. Exact solutions for composite laminates in cylindrical bending. J. Comp. Mats. 3, 398–411.

Reddy, J.N., 1989. On the generalization of displacement-based laminate theories. Applied Mechanics Reviews 42(11), 213–222.

Reddy, J. N., 1993. An evaluation of equivalent-single-layer and layerwise theories of composite laminates. Composite Structures 25(1), 21–35.

Reissner, E., 1944. On the theory of bending of elastic plates. Journal of Mathematics and Physics 23, 184–191.

Shupikov, A.N., Ugrimov, S.V., 1997. High-order theory for multilayer plates. Current problems of strength and optimization of structures and machines, pp. 103–109.

Smetankina, N.V., Sotrikhin, S. Yu., Shupikov, A.N., 1995. Theoretical and experimental investigation of vibration of multilayer plates under the action of impulse and impact loads. International Journal of Solids and Structures 32(8/9), 1247–1258.